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Inequalities for Differences of Powers

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1. INTRODUCTION

This paper is inspired by the works of Leach and Sholander [6, 7] who investigated the comparison problem of the so-called extended mean values $E_{r,s}$ defined by the formula

$$E_{r,s}(x, y) = E(r, s; x, y) = \left(\frac{x^s - y^s}{s} \cdot \frac{r}{x^r - y^r}\right)^{1/(s-r)} \tag{1}$$

for $r, s \in \mathbb{R}$, $x, y > 0$ with $rs(r-s)(x-y) \neq 0$.

Stolarsky [14] was the first to define E and he showed that E can be extended to be continuous on the domain

$$\{(r, s; x, y): r, s \in \mathbb{R}, x, y > 0\}.$$

Most of the classical two variable means are special cases of E, for instance, $E_{1,2} = A$ is the arithmetic mean, $E_{0,0} = G$ is the geometric mean, $E_{-2,-1} = H$ is the harmonic mean, and more generally, the rth power mean is equal to $E_{r,2r}$. The study of the so-called logarithmic mean $L = E_{0,1}$ and identric mean $I = E_{1,1}$ has also a rich literature (see Carlson [3], Dodd [5], Lin [8], Burk [2], Pittinger [12, 13], Székely [15], and Brenner [1]).

In [7] Leach and Sholander solved the problem of comparison of the above means E; that is, they found necessary and sufficient conditions for the parameters r, s, u, v in order that

$$E_{r,s}(x, y) \leqslant E_{u,v}(x, y) \tag{2}$$

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be satisfied for all positive x and y. If $r \neq s$ and $u \neq v$ then we rearrange (2) to obtain

$$1 \leqslant \left|\frac{x^r - y^r}{r}\right|^{1/(s-r)} \left|\frac{x^s - y^s}{s}\right|^{1/(r-s)} \left|\frac{x^u - y^u}{u}\right|^{1/(u-v)} \left|\frac{x^v - y^v}{v}\right|^{1/(v-u)} \tag{3}$$

The aim of the present note is to investigate the following more general inequality

$$1 \leqq \left|\frac{x^{a_1} - y^{a_1}}{a_1}\right|^{\alpha_1} \cdots \left|\frac{x^{a_k} - y^{a_k}}{a_k}\right|^{\alpha_k} \tag{4}$$

with the additional assumption

$$0 = \alpha_1 + \cdots + \alpha_k. \tag{5}$$

It is easy to see that (4) is more general than (3) even in the case $k = 4$.

In Section 2 we derive general necessary conditions in order that (4) be valid for all x, $y > 0$ with $x \neq y$. In Section 3 we prove that the necessary conditions obtained are also sufficient if we assume several additional conditions on the parameters $a_1, ..., a_k$ and k.

2. Necessary Conditions

Let us adopt the following convention: *If* $a = 0$ *then* $(x^a - y^a)/a$ *stands for* $\ln x - \ln y$ *for all* x, $y > 0$.

Using this convention we can allow the parameters $a_1, ..., a_k$ to be zero in equality (4).

THEOREM 1. *Let k be a natural number and $a_1, ..., a_k$, $\alpha_1, ..., \alpha_k$ be real numbers with* (5). *Then in order that* (4) *be valid for all different positive x and y it is necessary that the following three conditions be satisfied*

(i) $0 = \alpha_1 a_1 + \cdots + \alpha_k a_k$,

(ii) $0 = \alpha_1 a_1^2 + \cdots + \alpha_k a_k^2$,

(iii) $0 \leqq \alpha_1 f(a_1) + \cdots + \alpha_k f(a_k)$,

where

$$f(x) = -\ln |x| \qquad \textit{for} \quad x \neq 0$$

if either $0<\min a_i$ *or* $\max a_i<0$;

$$f(x)=1 \qquad \textit{for} \quad x=0,$$
$$=0 \qquad \textit{for} \quad x\neq 0$$

if either $0=\min a_i$ *or* $\max a_i=0$; *and finally*

$$f(x)=|x| \qquad \textit{for all real } x$$

if $\min a_i<0<\max a_i$.

Proof. To prove (i) put tx and ty into (4) instead of x and y, respectively. Then we get

$$1\leqq t^{\alpha_1 a_1+\cdots+\alpha_k a_k}\left|\frac{x^{a_1}-y^{a_1}}{a_1}\right|^{\alpha_1}\cdots\left|\frac{x^{a_k}-y^{a_k}}{a_k}\right|^{\alpha_k}$$

for all positive t. This inequality shows that the function

$$t\to t^{\alpha_1 a_1+\cdots+\alpha_k a_k}$$

has a positive greatest lower bound. Therefore we have (i).

The proof of (ii) is a bit longer. Put $x=e^s$ and $y=e^{-s}$ into (4). Then, applying (5), we easily obtain

$$0\leqq\alpha_1 g(a_1 s)+\cdots+\alpha_k g(a_k s), \tag{6}$$

where $g(0)=0$ and, for $x\neq 0$,

$$g(x)=\ln(\sinh x/x)=\ln(1+x^2/3!+x^4/5!+x^6/7!+\cdots)$$
$$=x^2/6-x^4/180+x^6/2835-\cdots.$$

Multiplying (6) by $6/s^2$ and taking the limit $s\to 0$, we get (ii).

To prove the necessity of condition (iii) we have to distingish three cases.

Case I. Either $0<\min a_i$ or $\max a_i<0$. Let $y=1$ in (4) and assume that $a_i>0$ for all i (resp. $a_i<0$). Calculating the limit of the right-hand side of (4) if x tends to zero (resp. to infinity) we easily see that

$$1\leqq|a_1|^{-\alpha_1}\cdots|a_k|^{-\alpha_k}$$

which is equivalent to (iii) in this case.

Case II. Either $0=\min a_i$ or $\max a_i=0$. Denote

$$I=\{i\colon 1\leqq i\leqq k,\ a_i\neq 0\}, \qquad I_0=\{i\colon 1\leqq i\leqq k,\ a_i=0\}.$$

Using these notations, (4) turns into

$$\prod_{i\in I}\left|\frac{x^{a_i}-y^{a_i}}{a_i}\right|^{-\alpha_i} \leqq \prod_{i\in I_0}\left|\frac{x^{a_i}-y^{a_i}}{a_i}\right|^{\alpha_i} = |\ln(x/y)|^{\sum_{i\in I_0}\alpha_i}. \tag{7}$$

Let $y=1$ in this inequality and assume that $a_i \geqq 0$ for all i (resp. $a_i \leqq 0$). Then taking the limit $x \to 0$ (resp. $x \to \infty$) we find that the left-hand side of (7) tends to the positive value

$$\prod_{i\in I} |a_i|^{\alpha_i}.$$

However, if $\sum_{i\in I_0}\alpha_i$ were negative then the right-hand side of (7) would tend to zero. Therefore $\sum_{i\in I_0}\alpha_i$ must be nonnegative, i.e., (iii) is valid.

Case III. $\min a_i < 0 < \max a_i$. As we have seen, (4) implies (6) for all $s \neq 0$. Multiply (6) by $1/s$ and calculate the limit as $s \to \infty$. By L'Hosital's rule we have

$$\lim_{s\to\infty} g(a_i s)/s = \lim_{s\to\infty} (a_i \coth(a_i s) - 1/s) = |a_i|;$$

therefore it follows from (6) that

$$0 \leqq \alpha_1 |a_1| + \cdots + \alpha_k |a_k|,$$

which completes the proof of the theorem.

3. Sufficient Conditions

First we need two lemmas.

Lemma 1. *If x is a positive value then*

$$\sinh x < x \cosh x \tag{8}$$

and

$$x^3 \cosh x < \sinh^3 x. \tag{9}$$

Proof. Expanding in Maclaurin series and comparing the coefficient of x^{2n+1} on the right- and left-hand sides, we find that (8) is valid.

To prove (9), we apply the "multiple angle" formula

$$\sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x.$$

Then expanding both sides of (9) in Maclaurin series and comparing the coefficients of x^{2n+1}, we can see that it is enough to show that

$$4(2n-1)\,2n(2n+1)+3 \leqq 3^{2n+1}$$

holds for $n=1, 2, \ldots$. This inequality holds with equality in the case $n=1$ and $n=2$ and hen the right-hand side increases more rapidly than the left-hand side, thus (9) is valid.

LEMMA 2. *Let* $a_1 \leqq a_2 \leqq a_3 \leqq a_4$ *be arbitrary with* $0 \leqq a_1 + a_4$ *and* $0 \leqq a_2 + a_3$. *Then there exist real* a, b *and nonnegative* c, d *such that*

$$g(a_i) = a + ba_i + ca_i^2 + df(a_i) \tag{10}$$

holds for $i = 1, 2, 3, 4$. (f *and* g *are defined in condition* (iii) *and in the proof of Theorem* 1, *respectively.*)

Proof. First we assume that $a_1 < a_2 < a_3 < a_4$. In the proof of the lemma we shall distinguish five cases.

Case I. $0 < a_1$. Then $f(x) = -\ln x$. We prove that the system of linear equations (10) is solvable for a, b, c, d, or, equivalently,

$$D = \det |1, a_i, a_i^2, \ln a_i|_{i=1}^4 \neq 0.$$

As is known from approximation theory, there exists a Lagrange interpolation polynomial $P(x) = p_0 x^3 + p_1 x^2 + p_2 x + p_3$ such that

$$\ln a_i = P(a_i) \qquad (i = 1, 2, 3, 4).$$

This means that $h(x) = \ln x - P(x)$ vanishes at four different points. A repeated application of Rolle's theorem shows that $h'''(x_0) = 0$ for some positive x_0, i.e.,

$$2/x_0^3 = 6p_0.$$

Therefore, $p_0 \neq 0$ and, for D, we have

$$D = \det |1, a_i, a_i^2, \ln a_i|_{i=1}^4 = \det |1, a_i, a_i^2, P(a_i)|_{i=1}^4$$
$$= p_0 \det |1, a_i, a_i^2, a_i^3|_{i=1}^4 = p_0 \prod_{1 \leqq i < j \leqq 4} (a_i - a_j) \neq 0.$$

Thus a, b, c, d exist such that (10) is satisfied. We want to prove that c and d are nonnegative. Let

$$h(x) = g(x) - a - bx - cx^2 + d \ln x.$$

Then, by (10), $h(a_i)=0$ for $i=1, 2, 3, 4$. Applying Rolle's theorem, we find that $h'''(x)$ and $(x^2h''(x))'$ vanish for some positive $x=x_1$ and $x=x_2$, i.e.,

$$x_1^3 g'''(x_1)=2d \tag{11}$$

and

$$2g''(x_2)+x_2 g'''(x_2)=4c, \tag{12}$$

respectively. However, by Lemma 1,

$$g'''(x)=2(x^3 \cosh x-\sinh^3 x)/(x \sinh x)^3<0 \tag{13}$$

and

$$2g''(x)+xg'''(x)=2(x \cosh x-\sinh x)/\sinh^3 x>0$$

for $x>0$. Therefore it follows from (11) and (12) that c and d are positive.

Case II. $a_1=0$. Then $f(a_1)=1$ and $f(a_i)=0$ for $i=2, 3, 4$. Thus (10) reduces to the following system of equations

$$0=a+d, \tag{14}$$

$$g(a_i)=a+ba_i+ca_i^2 \qquad (i=2, 3, 4). \tag{15}$$

The determinant of the system (15) is nonzero, therefore a, b, c, d exist satisfying (14) and (15). Instead of $d \geqq 0$ it is enough to prove that $a \leqq 0$. Let

$$h(x)=g(x)-a-bx-cx^2.$$

Then, by (15), h vanishes at a_2, a_3, and a_4 therefore $h''(x)$ and $(h(x)/x)''$ also vanish for some positive $x=x_1$ and $x=x_2$, i.e.,

$$g''(x_1)=2c \tag{16}$$

and

$$x_2^2 g''(x_2)-2x_2 g'(x_2)+2g(x_2)=2a, \tag{17}$$

respectively. Since

$$g''(x)=(\sinh^2 x-x^2)/(x \sinh x)^2>0, \tag{18}$$

hence (16) implies $c>0$. To prove that $a \leqq 0$, it is enough to show that

$$k(x)=x^2g''(x)-2xg'(x)+2g(x)<0 \tag{19}$$

for $x>0$. Obviously, $k(0)=0$ and $k'(x)=x^2g'''(x)$, which is negative by (13). Therefore k is strictly decreasing, whence $0=k(0)>k(x_2)$ follows.

Case III. $a_1 < 0 \leqq a_2$. Then $f(x) = |x|$ and (10) reduces to the system

$$g(a_1) = a + (b-d)\,a_1 + ca_1^2,$$
$$g(a_i) = a + (b+d)\,a_i + ca_i^2 \qquad (i = 2, 3, 4).$$

It is easy to check that this system has a unique solution a, b, c, d. Let

$$h(x) = g(x) - a - (b+d)\,x - cx^2.$$

Supposing that $a_2 \neq 0$, the same argument that was used in Case II leads to the inequalities $a < 0$ and $c = \frac{1}{2} g''(x_1) > 0$ with a suitable $a_2 < x_1 < a_4$. Define

$$h_1(x) = g(x) - cx^2 = g(x) - \tfrac{1}{2} g''(x_1)\,x^2$$

and

$$h_2(x) = a + (b+d)\,x.$$

We obviously have $h_1(a_i) = h_2(a_i)$ for $i = 2, 3, 4$.

Since g''' is negative, hence g'' is strictly decreasing on the interval $(0, \infty)$. Therefore $h_1''(x) = g''(x) - g''(x_1)$ is positive (resp. negative) if $0 < x < x_1$ (resp. $x_1 < x$), i.e., h_1 is convex on $(0, x_1)$ and concave on (x_1, ∞). Since $h_1'(0) = h_1(0) = 0$, hence h_1 is a strictly positive, increasing, and convex function on the interval $(0, x_1)$. The concavity of h_1 on (x_1, ∞) and $h_1(x_1) > 0$ implies that h_1 vanishes at most one point in (x_1, ∞). On the other hand, $h_1(x)$ must be zero for a suitable $x = x_2$, since $\lim_{x \to \infty} h_1(x)/x^2 = -c < 0$; that is, $h_1(x)$ is negative if x is large enough.

For the function h_2 we have $h_2(0) = a < 0$ and $h_2(a_2) = h_1(a_2) > 0$ (since $0 < a_2 < x_1$), therefore h_2 is an increasing function. This means that $h_2(a_4) = h_1(a_4) > 0$. Thus $a_4 < x_2$. Now the assumptions $0 \leqq a_1 + a_4$ and $a_1 < 0$ imply $0 < -a_1 \leqq x_2$, whence it follows that

$$h_1(a_1) = h_1(-a_1) \geqq 0.$$

On the other hand, since h_2 is increasing, hence $h_2(a_1) < h_2(0) = a < 0$. Thus $h(a_1) = h_1(a_1) - h_2(a_1) > 0$; that is,

$$a + (b-d)\,a_1 + ca_1^2 = g(a_1) > a + (b+d)\,a_1 + ca_1^2,$$

whence we get $d > 0$.

To prove the assertion in the case $a_2 = 0$, it is enough to notice that the solutions a, b, c, d depend continuously on a_1, a_2, a_3, and a_4. Therefore, taking the limit $a_2 \downarrow 0$, we obtain $c, d \geqq 0$.

Case IV. $a_2 < 0 \leqq a_3$ and $a_1 a_2 < a_3 a_4$. (We remark that the assumptions $0 \leqq a_1 + a_4$ and $0 \leqq a_2 + a_3$ imply $0 < a_3$ and $a_1 a_2 \leqq a_3 a_4$, but we have excluded equality in this latter inequality.)

Now (10) reduces to the following system

$$g(a_i) = a + (b-d)\, a_i + c a_1^2 \qquad (i = 1, 2), \tag{20}$$

$$g(a_i) = a + (b+d)\, a_i + c a_1^2 \qquad (i = 3, 4). \tag{21}$$

It follows from (20) and (21) that

$$(a_2\, g(a_1) - a_1\, g(a_2))/(a_1 - a_2) = -a + c a_1 a_2$$

and

$$(a_4\, g(a_3) - a_3\, g(a_4))/(a_3 - a_4) = -a + c a_3 a_4,$$

respectively. Thus, setting $\hat{a}_3 = -a_2$ and $\hat{a}_4 = -a_1$, we obtain

$$c = \frac{1}{a_3 a_4 - a_1 a_2} \left(\frac{u(1/\hat{a}_3) - u(1/\hat{a}_4)}{1/\hat{a}_3 - 1/\hat{a}_4} - \frac{u(1/a_3) - u(1/a_4)}{1/a_3 - 1/a_4} \right)$$

and

$$a = \frac{a_1 a_2 a_3 a_4}{a_3 a_4 - a_1 a_2} \left(\frac{v(a_3) - v(a_4)}{a_3 - a_4} - \frac{v(\hat{a}_3) - v(\hat{a}_4)}{\hat{a}_3 - \hat{a}_4} \right),$$

where

$$u(x) = x g(1/x) \qquad \text{and} \qquad v(x) = g(x)/x$$

for $x > 0$. A simple calculation yields

$$u''(x) = g''(1/x)/x^3 > 0$$

(see (18)) and

$$v''(x) = (x^2 g''(x) - 2x g'(x) + 2g(x))/x^3 < 0$$

(see (19)); that is, u is convex and v is concave. Since $0 < \hat{a}_3 \leqq a_3$ and $0 < \hat{a}_4 \leqq a_4$, the convexity of u and the concavity of v implies that $c \geqq 0$ and $a \leqq 0$.

Concerning c we have two possibilities:

A. There exists $0 < x_1$ such that $c = \frac{1}{2} g''(x_1)$;

B. $c \geqq \sup \frac{1}{2} g''(x) = \frac{1}{2} g''(0) = \frac{1}{6}$.

If A holds then, as we have seen in the proof of Case III, $h_1(x) =$

$g(x)-\frac{1}{2}g''(x_1)\,x^2$ is concave for $x_1<x$ and there exists $x_1<x_2$ so that $\operatorname{sign} h_1(x)=\operatorname{sign}(x_2-x)$ for $x>0$. If B is valid then h_1 is negative and concave on $(0, \infty)$. Summarizing these possibilities, we can state that there exists $0<x_2$ so that $\operatorname{sign} h_1(x)=\operatorname{sign}(x_2-x)$ for $x>0$ and h_1 is concave for $x>x_2$.

Now we show that $b+d\geqq 0$. Assume on the contrary that $b+d<0$, then

$$h_2(x)=a+(b+d)\,x$$

takes only negative values for $x>0$, since it decreases and $h_2(0)=a\leqq 0$. On the other hand, by (21), $h_1(a_i)=h_2(a_i)$ for $i=3, 4$. Therefore $h_1(a_i)<0$, whence we get $x_2<a_i$ $(i=3, 4)$. Thus, by the concavity of h_1, $h_2(x_2)>h_1(x_2)=0$, which means a contradiction since $h_2(x_2)\leqq 0$.

A similar argument shows that $b-d\leqq 0$ is also valid. This inequality together with $b+d\geqq 0$ implies that $d\geqq |b|\geqq 0$, which was to be proved.

Case V. $a_1+a_4=0=a_2+a_3$. Then it is easy to check that $b=c=0$, $d=(g(a_4)-g(a_3))/(a_4-a_3)$, and $a=g(a_3)-da_3$ satisfies (20) and (21) and now $d\geqq 0$ and $c\geqq 0$ is obvious.

Thus we have proved the statement of the lemma in all of the possible cases provided that $a_1<a_2<a_3<a_4$.

If $\{a_1, a_2, a_3, a_4\}=\{\hat{a}_1, ..., \hat{a}_k\}$, where $\hat{a}_1<\cdots<\hat{a}_k$ and $k\leqq 3$, then choosing $\hat{a}_k<\hat{a}_{k+1}<\cdots<\hat{a}_4$ we obviously have $0\leqq\hat{a}_1+\hat{a}_4$, $0\leqq\hat{a}_2+\hat{a}_3$.

Now, applying the lemma for these values, it is easy to see that the resulting values a, b, c, d trivially satisfy (10). Thus the proof is complete.

Now we can state our main result on the sufficiency of conditions (i), (ii), and (iii).

THEOREM 2. *Let $a_1\leqq a_2\leqq a_3\leqq a_4$ be arbitrary real numbers with $(a_1+a_4)(a_2+a_3)\geqq 0$ and let $\alpha_1, ..., \alpha_4$ satisfy* (5) *for $k=4$. Then in order that* (4) *be valid for all positive x and y $(x\neq y)$ it is necessary and sufficient that conditions* (i), (ii), *and* (iii) *of Theorem* 1 *be satisfied for $k=4$.*

Proof. We have only to show the sufficiency of the conditions. As we have seen in the proof of Theorem 1, (4) implies (6) for all positive s. Putting $s=|\ln\sqrt{x/y}|$ into (6) and using condition (i), we can easily check that (4) also follows from (6). Therefore it is enough to prove that (6) is satisfied.

Because of symmetry we may assume that $0\leqq a_1+a_4$ and $0\leqq a_2+a_3$. Let s be fixed. If we apply Lemma 2 for the values $\hat{a}_i=a_is$ then we have the existence of real numbers $a=a(s)$, $b=b(s)$, $c=c(s)$, and $d=d(s)$ such that $c\geqq 0$, $d\geqq 0$, and

$$g(a_is)=a+ba_is+c(a_is)^2+df(a_is), \tag{22}$$

where f is the same function (independent of s) that was defined in Theorem 1. Mutliply (22) by α_i and add these equations to obtain

$$\sum_{i=1}^{4} \alpha_i g(a_i s) = a \sum_{i=1}^{4} \alpha_i + bs \sum_{i=1}^{4} \alpha_i a_i + cs^2 \sum_{i=1}^{4} \alpha_i a_i^2 + d \sum_{i=1}^{4} \alpha_i f(a_i s).$$

Now, applying (5), (i), (ii), (iii), $c \geqq 0$, and $d \geqq 0$ we see that each term on the right-hand side is nonnegative, whence we get (6) (for $k=4$) and Theorem 2 is proved.

4. Application

Using Theorem 2 in inequality (2), we obtain the result of Leach and Sholander [7].

Corollary. *Let r, s, u, v be arbitrary with $r \neq s$, $u \neq v$. Then (2) is satisfied for all x, $y > 0$ if and only if*

$$r + s \leqq u + v \tag{23}$$

and

$$e(r, s) \leqq e(u, v) \tag{24}$$

where

$$\begin{aligned} e(x, y) &= (x-y)/\ln(x/y) \qquad &&\textit{for} \quad xy > 0, x \neq y, \\ &= 0 &&\textit{for} \quad xy = 0 \end{aligned}$$

if either $0 \leqq \min(r, s, u, v)$ *or* $\max(r, s, u, v) \leqq 0$; *and*

$$e(x, y) = (|x| - |y|)/x - y) \qquad \textit{for} \quad x, y \in \mathbb{R}, x \neq y$$

if $\min(r, s, u, v) < 0 < \max(r, s, u, v)$.

Proof. (2) is equivalent to (3). By Theorem 1, in order that (3) be valid it is necessary that (i), (ii), and (iii) be satisfied with

$$\alpha_1 = 1/(s-r), \qquad \alpha_2 = 1/(r-s), \qquad \alpha_3 = 1/(u-v), \qquad \alpha_4 = 1/(v-u)$$

and

$$a_1 = r, \qquad a_2 = s, \qquad a_3 = u, \qquad a_4 = v.$$

Now (i) is obvious. It is easy to check that (ii) and (iii) turn into (23) and (24), respectively.

We have to show that (23) and (24) are also sufficient conditions. Because of symmetry we may assume that $r<s$ and $u<v$. By (23), only the following possibilities can occur:

$$r<s\leqq u<v, \qquad r\leqq u<v\leqq s, \qquad r\leqq u\leqq s\leqq v, \qquad u\leqq r<s\leqq v.$$

If $0\leqq r+s\leqq u+v$, then denoting r, s, u, v by a_1, a_2, a_3, a_4 so that $a_1\leqq a_2\leqq a_3\leqq a_4$, we can see that in each of the above possibilities $0\leqq a_1+a_4$ and $0\leqq a_2+a_3$ holds therefore Theorem 2 can be applied and it shows that (23) and (24) are sufficient conditions.

The case $r+s\leqq u+v\leqq 0$ is completely similar.

If $r+s\leqq 0\leqq u+v$ then it is easy to show that

$$E_{r,s}(x, y)\leqq G(x, y)\leqq E_{u,v}(x, y),$$

where G denotes the geometric mean. Thus (2) holds trivially and the proof is complete.

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